Normalization in the Dual Calculus with Sigma Reductions

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A proof that call-by-value dual calculus is strongly normalizing using a novel logical-predicates argument.

Syntax

Types
$A, B \quad ::= \quad X \mid A \wedge A \mid A \vee A \mid \neg A$

Terms

m, n	$::=$	x	Variable
	$\mid$	$\langle m, n\rangle$	Tuple
	$\mid$	$[k]$not	Negation
	$\mid$	$\langle m\rangle$inl $\mid$ $\langle n\rangle$inr	Disjoint Union
	$\mid$	$(S).\alpha$	Let/cc

Coterms

k, l	$::=$	α	Covariable
	$\mid$	fst$[k]$ $\mid$ snd$[l]$	Projection
	$\mid$	not$\langle m\rangle$	Negation
	$\mid$	$[k, l]$	Pattern Match
	$\mid$	$x.(S)$	Let/ce

Statements

S	$::=$	$m \bullet k$	Statement

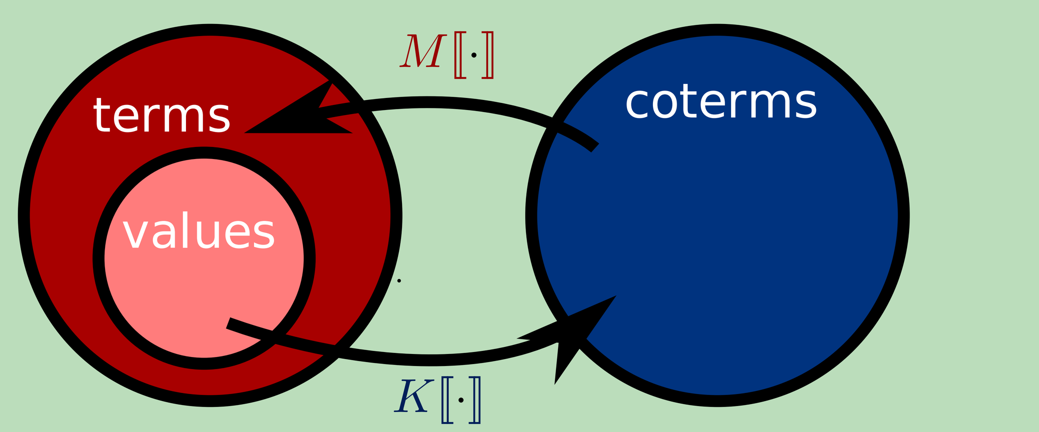
The Logical Relation

$S \in S[\![\#]\!]$ iff $S \to^* S' \not\to$

$x \in V[\![A]\!]$ (always)
$\langle v, w\rangle \in V[\![A \wedge B]\!]$ iff $v \in V[\![A]\!]$ and $w \in V[\![B]\!]$
$\langle v\rangle$inl $\in V[\![A \vee B]\!]$ iff $v \in V[\![A]\!]$
$\langle w\rangle$inr $\in V[\![A \vee B]\!]$ iff $w \in V[\![B]\!]$
$[k]$not $\in V[\![\neg A]\!]$ iff $k \in K[\![A]\!]$

$k \in K[\![A]\!]$ iff for all $v \in V[\![A]\!], v \bullet k \in S[\![\#]\!]$

$m \in M[\![A]\!]$ iff for all $k \in K[\![A]\!], m \bullet k \in S[\![\#]\!]$



Typing Judgments

$\Gamma \vdash m\colon A; \Theta$

$\Gamma; k\colon A \vdash \Theta$

$S\colon (\Gamma \vdash \Theta)$

Γ Logical premises/ Computational inputs

Θ Possible conclusions/ Output channels

Strong Normalization

Definition. Substitution σ is Γ, Θ-logical when

(i) for all $x \in \mathrm{dom}(\sigma)$, $\sigma(x) \in V[\![\Gamma(x)]\!]$, and

(ii) for all $\alpha \in \mathrm{dom}(\sigma)$, $\sigma(\alpha) \in K[\![\Theta(\alpha)]\!]$.

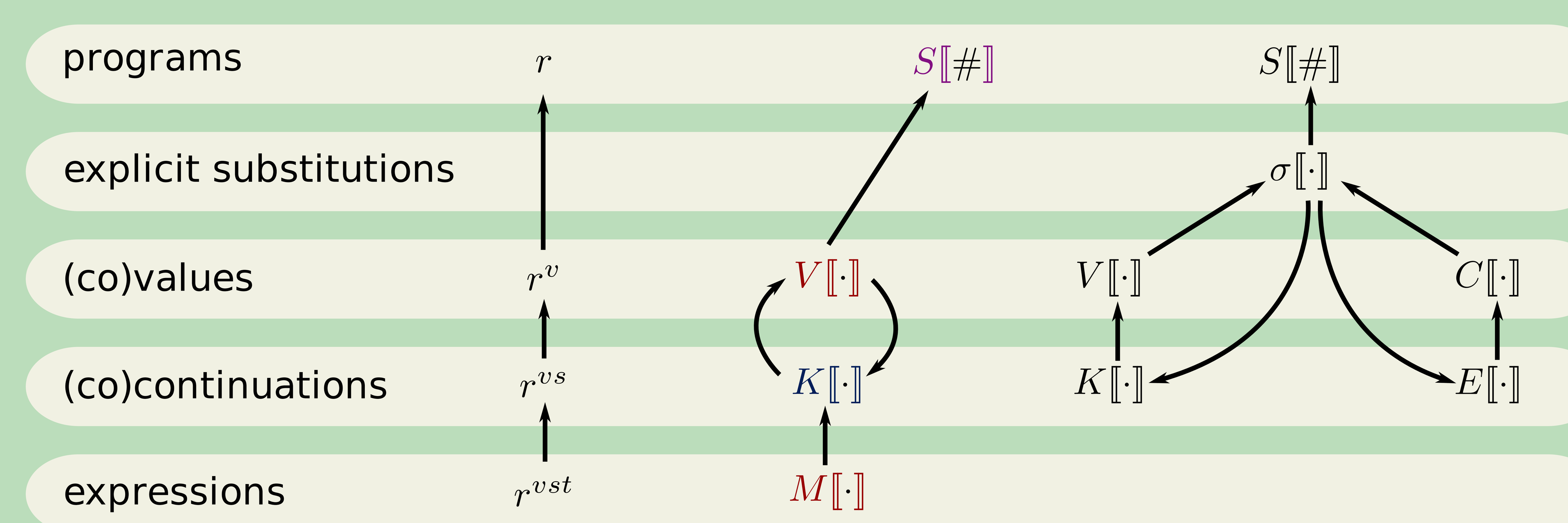
Lemma. *Suppose $\mathcal{D}$ is a typing derivation, then*

- *if $\mathcal{D} :: S\colon (\Gamma \vdash \Theta)$ then for all Γ, Θ-logical σ, $\sigma(S) \in S[\![\#]\!]$,*
- *if $\mathcal{D} :: \Gamma \vdash v\colon A; \Theta$ then for all Γ, Θ-logical σ, $\sigma(v) \in V[\![A]\!]$,*
- *if $\mathcal{D} :: \Gamma \vdash m\colon A; \Theta$ then for all Γ, Θ-logical σ, $\sigma(m) \in M[\![A]\!]$,*
- *if $\mathcal{D} :: \Gamma; k\colon A \vdash \Theta$ then for all Γ, Θ-logical σ, $\sigma(k) \in K[\![A]\!]$.*

Corollary. *Every well-typed, call-by-value dual calculus statement is strongly normalizing.*

Open Questions: How does this proof strategy relate to...

...Top-Top closure? ... the calculus of unity?

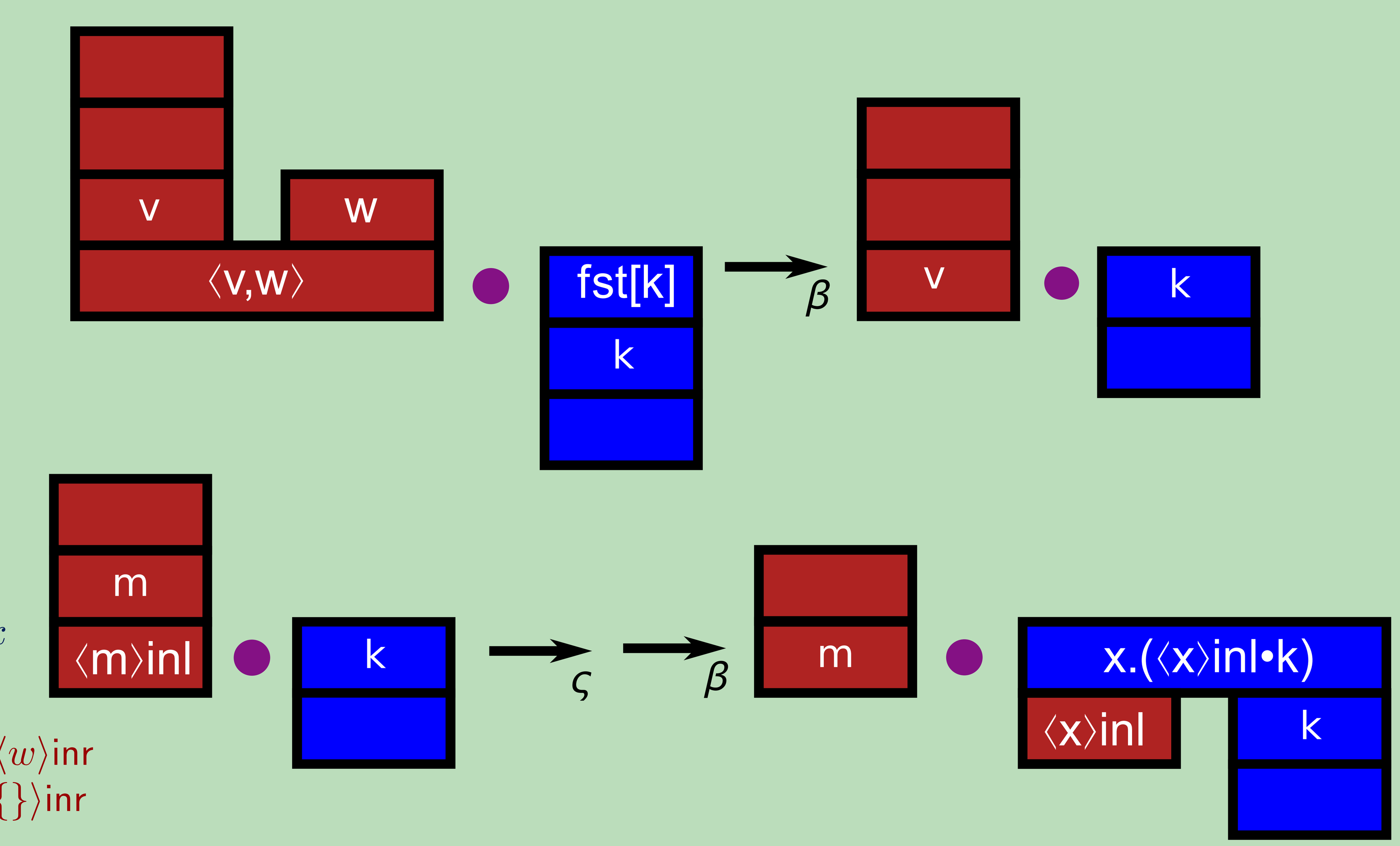


Call-by-value Reduction

$\langle v, w\rangle \bullet \mathsf{fst}[k] \to_\beta v \bullet k$
$\langle v, w\rangle \bullet \mathsf{snd}[l] \to_\beta w \bullet l$
$\langle v\rangle\mathsf{inl} \bullet [k, l] \to_\beta v \bullet k$
$\langle w\rangle\mathsf{inr} \bullet [k, l] \to_\beta w \bullet l$
$v \bullet x.(S) \to_\beta \{v/x\}S$
$(S).\alpha \bullet k \to_\beta \{k/\alpha\}S$
$[k]\mathsf{not} \bullet \mathsf{not}\langle m\rangle \to_\beta m \bullet k$

$E\{m\} \bullet k \to_\varsigma$
$(m \bullet x.(E\{x\} \bullet \beta)).\beta \bullet k$

$v, w \quad ::= \quad x \mid \langle v, w\rangle \mid [k]\mathsf{not} \mid \langle v\rangle\mathsf{inl} \mid \langle w\rangle\mathsf{inr}$
$E \quad ::= \quad \langle\{\}, n\rangle \mid \langle v, \{\}\rangle \mid \langle\{\}\rangle\mathsf{inl} \mid \langle\{\}\rangle\mathsf{inr}$



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